DISTRIBUTION FUNCTIONS
NOMENCLATURE & DEFINITIONS

① DISTRIBUTION FUNCTION

PHASE SPACE DENSITY $f(\vec{r},\vec{v})$. Most often written as $f(\vec{v})$.

Usually normalized such that $\int f(\vec{v})\,d^3v = N$ (Particle Density)

Units of $f(\vec{r},\vec{v})$ are $\dfrac{1}{cm^3\left(\frac{cm}{s}\right)^3}$ or $\dfrac{\overset{\text{unitless}}{\text{particles}}}{cm^3\left(\frac{cm}{s}\right)^3}$. Thus $dN = f\,d^3v$

② (Particle) Flux: $\vec{J} = \overset{\text{fluid velocity, or bulk velocity}}{\vec{v}}\,N = \int \vec{v}\,dN$. Units: $\dfrac{1}{cm^2\,s}$ or $\dfrac{\#\text{ particles}}{cm^2\,s}$

$$d\vec{J} = \vec{v}\,dN = \vec{v} f\cdot d^3v$$

Differential (Particle) Flux: (A) $\dfrac{d\vec{J}}{dE}$ units: $\dfrac{1}{cm^2\,s\,keV}$ or $\dfrac{\#\text{ particles}}{cm^2\,s\,keV}$

Differential (Particle) Flux: (B) $\dfrac{d\vec{J}}{dE\,d\Omega}$ units $\dfrac{1}{cm^2\,s\,keV\,str}$ or $\dfrac{\#\text{ particles}}{cm^2\,s\,keV\,str}$

str: steradian

③ Energy Flux: $\vec{Q} = \int \vec{v}\,dN\cdot E$ Units: $\dfrac{keV}{cm^3\,(s/cm)} = \dfrac{keV}{cm^2\cdot s}$

$$d\vec{Q} = \vec{v}\,dN\cdot E = \vec{v}E\,f\,d^3v$$

Differential Energy Flux: (A) $\dfrac{d\vec{Q}}{dE}$ units: $\dfrac{keV}{cm^2\cdot s\cdot keV}$

Differential Energy Flux: (B) $\dfrac{d\vec{Q}}{dE\,d\Omega}$ Units: $\dfrac{keV}{cm^2\,s\,keV\,str}$

distribution function

From (e.g. Maxwellian) to counts

$f \ (cm^6/s)^{-1}$

$$f(\vec{r},\vec{v}) = \frac{N}{\sqrt{\pi^3 v_{th}^3}} e^{-\frac{(\vec{v}-\vec{v}_0)^2}{v_{th}^2}} \qquad v_{th} = \sqrt{\frac{2kT}{m}}$$

Note: $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$ Such that $\int f d^3v = N$

Given $N, \vec{v}_0, T$ can get f vs. v, ϑ, φ
and f vs. E, ϑ, φ

Diff Flux $\frac{d\vec{J}}{dE d\Omega}$ $(cm^2 s \ keV \ str)^{-1}$

$$d\vec{J} = \vec{v} f(v) d^3v$$

$$d\vec{J} = \vec{v} f(E) d^3v = \vec{v} f(E,\vartheta) v^2 d\Omega dv = v^3 \hat{v} f(E) d\Omega dv =$$

$$= v^2 \hat{v} f(E) d\Omega \, d\left(\frac{v^2}{2}\right)$$

$$= \frac{2}{m^2}\left(\frac{mv^2}{2}\right) \hat{v} f(E,\vartheta) d\Omega \, d\left(\frac{mv^2}{2}\right) = \frac{2}{m^2} E \hat{v} f(E) d\Omega dE$$

$$\frac{d\vec{J}}{dE d\Omega} = \frac{2\hat{v}}{m^2} E f(E) = \frac{2}{m^2} \vec{E} f(\vec{E}) \qquad \vec{E} \equiv (E, \vartheta, \varphi)$$

Given $N, \vec{v}_0, T$ can get $dJ/dEd\Omega$ vs. E, ϑ, φ

Diff. Energy Flux $\frac{d\vec{Q}}{dE d\Omega}$ $\left(\frac{keV}{cm^2 \, s \, keV \, str}\right)$

$$d\vec{Q} = \vec{v} E f(\vec{v}) d^3v = \vec{v} \vec{E} f(\vec{E}) v^2 dv d\Omega = v \cdot \vec{E} f(\vec{E}) v \, d\left(\frac{v^2}{2}\right) d\Omega =$$

$$= 2 \frac{v^2}{2} \vec{E} f(\vec{E}) \, d\left(v^2/2\right) d\Omega = \frac{2}{m^2} E \cdot \vec{E} f(\vec{E}) dE d\Omega$$

$$\frac{d\vec{Q}}{dE d\Omega} = \frac{2}{m^2} E \vec{E} f(\vec{E}) = E \cdot \left(\frac{d\vec{J}}{dE \cdot d\Omega}\right)$$

(4) Particle Count Rate

$$\vec{R}\left(\overset{\text{units}}{s^{-1}}\right) = \frac{d\bar{J}}{dE\,d\underline{o}} * G * E \cdot \mathcal{E} = \frac{d\bar{Q}}{dE\,d\Omega} \cdot G \cdot \mathcal{E}$$

⇒ Geometric factor for Electrostatic Analyzers

$$G = \left[\underset{\text{cm}^2}{\text{Area}} \cdot \underset{\text{str}}{\text{Solid Angle}} \cdot \underset{\text{eV}}{\text{Bin Energy width}} \Big/ \underset{\text{eV}}{\text{energy of channel}}\right]$$

units: $\text{cm}^2\ \text{str eV}/\text{eV}$

Multiplied by units of $\frac{d\bar{Q}}{dE\,d\Omega}\left(\frac{\text{eV}}{\text{cm}^2\ \text{s sr eV}}\right)$ results in $(\#/s)$

G is a function of energy channel, as well as detector direction. Thus $G(E, \theta, \varphi)$ depends on specifics of instrument.

→ ΔE = energy bandwidth

For electrostatic analyzers the inherent energy resolution δE is proportional to the energy: $E = \frac{q \cdot V}{2} \cdot \frac{R}{\Delta}$ where Δ is gap, R is radius. This is because $\delta E = qV/2 \cdot \delta R/\Delta$ and $\frac{\delta E}{E} = \frac{\Delta}{R} = \frac{1}{k}$, where K is the analyzer constant. So $\Delta E/E$ is constant and is part of the geometric factor of the instrument.

→ $\mathcal{E}$ = detection efficiency. This is the number of counts registered for a given number of particles that hit the detector. This gets us from #particles/s to #counts/s. Typical values for electrostatic analyzers is 0.6 – 0.7

(5) Particle Counts.

$$\vec{C}\,(\text{unitless}) = \bar{R} * \tau$$

τ is the measurement interval (seconds)

<u>Note</u>: $\dfrac{\bar{R}}{G\,\varepsilon} = \dfrac{d\bar{Q}}{dE\,d\Omega} = E\,\dfrac{d\bar{J}}{dE\,d\Omega} \propto E^2 \cdot f$

diff. energy flux — diff. flux — distr. function

<u>UNITS</u>: $\bar{R}\left(\#\,\text{counts}/s\right)$; $\dfrac{d\bar{Q}}{dE\,d\Omega}\left(\dfrac{eV}{eV\,cm^2\,s\,str}\right)$; $G\left(cm^2\,str\;eV/eV\right)$

<u>NOTE</u>: counts/s are <u>dead time corrected</u>

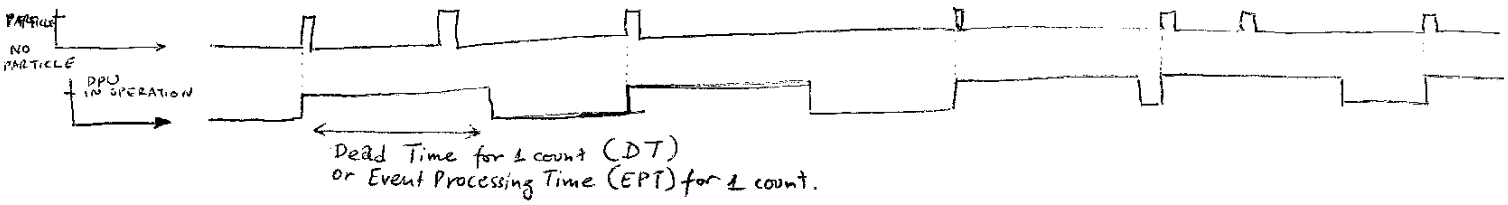
This dead-time correction is necessary before using the above equation.

Counts "C" is # counts per sample

$$C = R * AT$$, where AT is accumulation time

DEAD TIME CORRECTIONS

In addition to geometric factor, detector efficiency and energy bandwidth we need to apply a dead time correction to the counts measured to get the counts in the plasma.
The dead time results from the processing that needs to take place once an event has been registered, as the DPU is occupied.



Thus the <u>total dead time</u> is a function of the number of counts.
<u>Total $DT = C_m \cdot EPT$.</u> Live time (LT) is the time the detector was in operation. <u>$LT = AT - DT$</u>, wher AT is the accumulation interval, or Accumulation time. EPT = event processing time for 1 count.
Then the actual count rate is: $R_{real} = \dfrac{C_m}{LT} = \dfrac{C_m}{AT-DT} \Rightarrow$

$$\Rightarrow \quad R_{real} = \frac{C_m}{AT - C_m\, EPT} = \frac{C_m/AT}{1 - C_m \frac{EPT}{AT}} = \frac{R_{measured}}{1 - \underbrace{R_{measured} \cdot EPT}_{\text{always} < 1}}$$

: you cannot have in a given Δt more particles than EPT's

Thus dead time correction consists of dividing the measured rate by the quantity $(1 - R_{measured} \cdot EPT)$.

<u>NOTE</u>: $\dfrac{1}{EPT}$ = max count rate $\left(\text{because } \dfrac{1}{EPT} = \dfrac{AT}{EPT}\,\dfrac{1}{AT} = \dfrac{\text{max \# of counts}}{AT} = \text{max count rate}\right)$

FURTER READING: **Curtis et al.** "On board data analysis techniques..." Rev. Sci. Inst. 60, 372, 1989

For on-board processing on AMPTE/IRM the formula given on page 374 of D. Curtis's paper can be understood as follows:

For each sample

$$R_{real} = \frac{C_m}{AT - C_m\,EPT} = \frac{C_m/EPT}{\frac{AT}{EPT} - C_m}$$

* The C_m/EPT in the numerator can be normalized away and re-entered on the ground. Thus $CR_n(\circledcirc)$ represents the normalised count rate = C_m/EPT in units that maximize the dynamical range of the on-board computations
* $\left(\frac{AT}{EPT}\right)$ is the accumulation time over a single energy measurement divided by the event processing time (in paper EPT is ~~refered~~ to as "dead time") and represents the maximum number of counts (denoted as CR_{max} in paper)
* C_m is the measured counts refered to as CR_i in the paper.

Thus $R_{real} \overset{\text{proportional to...}}{\approx} \dfrac{CR_n}{CR_{max} - CR_i}$

FROM COUNTS TO DISTRIBUTION FUNCTION

from $C_{ijk} \xrightarrow{\text{to}} f_{ijk}$

E_i/V_i $\quad \theta_j \quad \varphi_k$

IRM-SPECIFIC

Assuming $\frac{\Delta E}{E} = \text{constant} = \frac{2\Delta v}{v}$

AT = Accumulation time $= \frac{\text{Spin Period}}{16 \times 32} = \frac{4}{512} = \frac{1}{128}\,s$

EPT = Event Processing Time $\sim 10^{-6}\,s$

R_m = measured count rate $= \frac{C_m}{AT}$

C_m = measured counts per sample

R_r = real count rate (dead time corrected)

G = Geometric factor (θ)

ΔE = Energy bandwidth (E)

ε = Efficiency (E, θ)

$$R_m = \frac{C_m}{AT}$$

$$R_r = \frac{R_m}{1 - R_m \cdot EPT}$$

$$R_r = \frac{dJ}{dE\,d\Omega} G \cdot E \cdot \varepsilon = \frac{2}{m^2} E \cdot f \cdot G \cdot \frac{\Delta E}{E} \cdot E \cdot \varepsilon = 2\frac{E^2}{m^2} f \cdot G \cdot \frac{2\Delta v}{v} \cdot \varepsilon = v^4 f(v) G \frac{\Delta v}{v} \varepsilon$$

According to p.3 $= \frac{2}{m^2} \vec{E} f(\vec{E})$

$$\Rightarrow \quad f(\vec{v}) = \frac{C_m/AT}{1 - \frac{C_m}{AT} \cdot EPT} \cdot \frac{1}{v^4} \cdot \frac{1}{\left[G\left(\frac{\Delta v}{v}\right)\varepsilon\right]}$$

Discrete Steps:

$$\boxed{f_{ijk} = \frac{(C_{ijk}/AT)}{1 - \frac{C_{ijk}}{AT} \cdot EPT} \cdot \frac{1}{V_i^4} \cdot \frac{1}{G_k\left(\frac{\Delta v}{v}\right)\varepsilon_{ik}}}$$

$\langle V_i^4 \rangle \qquad \left\langle \frac{\Delta v}{v} \right\rangle$

$$f_{ijk} = \frac{C_{ijk}/AT}{1 - \frac{C_{ijk}}{AT} EPT} \cdot \frac{1}{\frac{2E^2}{m^2}} \cdot \frac{1}{G_k \cdot \text{eff} \cdot \left(\frac{\Delta E}{E}\right)}$$