Kinetic Processes and MHD Models:
Can they work together?

Michael Hesse
NASA GSFC
Overview of things to come

MHD:
- What is it?
- What is it good for?
- What is it not so good for?

Beyond MHD:
- Other physical models
- Extensions to MHD
  - Anisotropic pressure
  - Reconnection physics

Summary
MHD: What is it?

Continuity

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \]

Momentum

\[ \rho \frac{\partial}{\partial t} \vec{v} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p + \vec{j} \times \vec{B} \]

(resistive) Ohm’s law

\[ \overline{E} + \vec{v} \times \vec{B} = 0 \quad (= \eta \vec{j}) \]

Equation of state

\[ \frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p = -\gamma p \nabla \cdot \vec{v} \quad (+ (\gamma - 1) \eta \vec{j}^2) \]

Ampere’s law

\[ \nabla \times \vec{B} = \mu_0 \vec{j} \]

Faraday’s law

\[ \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \]

Absence of monopoles

\[ \nabla \cdot \vec{B} = 0 \]
MHD: When is it safe?

- Slow time scales: $\tau >> \Omega_{ci}^{-1}$
  - no ion (or electron cyclotron effects)

- Pressure isotropy for total (ion + electron) pressure
  - can write total pressure force as gradient

- Neglect/approximate heat flux
  - use ideal/polytropic pressure law

- "Large" spatial scales: $L >> c/\omega_{pi}$
  - ignore Hall effects and electron pressure effects

- Two plasma species: Ions and electrons
  - obtain one, combined, momentum equation

- $m_e/m_i$ negligible
  - ignore ion pressure in Ohm’s law

- Slow typical velocities $v_t/c << 1$
  - ignore displacement current
How is MHD good for you?

- MHD describes large scale structure
- MHD describes large scale transport
- MHD describes dominant large scale communication (by Alfvén waves)
- MHD describes large scale current systems, by $\nabla \times \vec{B}$
- MHD is “simple” (although codes are not!) and “cheap”
What is MHD used for anyway?

… it is hard to argue with success

Siebert and Siscoe
Research: CDPS formation

Cold-dense plasma sheet formation

Li et al., 2005

Orioset et al., 2005
Reproducing S/C observations

Gombosi et al.
Reproducing field-aligned currents
Providing global descriptions and forecasts
It can also produce lively debates, e.g.,…

How long is the tail for northward IMF?

Why does the PC potential saturate?

Raeder et al, 1999

Gombosi et al, 2000

Siscoe et al, 2005
Concerns and worries: MHD limitations

- Single fluid model
- Magnetic flux transport is based on ions only
- No drift physics – no Hall term, simple energy equation
- Simple energy equation: isotropic pressure, no heat flux
- Dissipation, diffusion is (numerical) resistive (ad-hoc) only
- Reconnection may be slow only
What to do: Go beyond MHD?

- Change physical description
  - Hall-MHD
  - Multi-fluid Models
  - Kinetic Models
  - Hybrid, full PIC

- Modify MHD
  - Transport “Coefficients”
    - Anisotropic Pressure
    - Reconnection Physics
    - ...
Consider Alternative Descriptions

• Hall-MHD
  – include (some) drift physics, ion-electron decoupling

• Multi-fluid modeling
  – include (fluid) physics of additional ion species

• Hybrid modeling
  – include kinetic ion physics (e.g, shocks)

• Fully kinetic modeling
  – do it all (EM waves, reconnection…)

... for global magnetosphere
MHD: Computational “Simplicity”

Rough estimate: 3D Magnetosphere

Size: $200R_E \times 60R_E \times 60R_E$
Cell size: $dx = 0.25R_E$
Without AMR ~ $4.6 \times 10^7$ cells
Eight MHD state variables
~$4 \times 10^8$ reals, 2-4G bytes

Ignored AMR, structured meshes etc.

Fastest wave mode is fast mode $v_f = (v_A^2 + v_S^2)^{1/2}$
Time step limitation is $dt < dx/v_f$
Step up: Hall-MHD models

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \]

\[ \rho \frac{\partial}{\partial t} \vec{v} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p + \vec{j} \times \vec{B} \]

\[ \vec{E} + \vec{v} \times \vec{B} = \eta \vec{j} + \frac{1}{en} \vec{j} \times \vec{B} - \frac{1}{en} \nabla p_e \]

\[ \frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p = -\gamma p \nabla \cdot \vec{v} \left( + (\gamma - 1) \eta j^2 \right) \]

\[ \nabla \times \vec{B} = \mu_0 \vec{j} \]

\[ \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \]

\[ \nabla \cdot \vec{B} = 0 \]

+ eqns of state for \( p_e \)

Continuity

Momentum

Ohm’s law

Equation of state

Ampere’s law

Faraday’s law

Absence of monopoles
Time step killer: Whistler waves

Electron MHD part of Hall-MHD Ohm’s law:

\[
\vec{E} = \frac{1}{en} \vec{j} \times \vec{B} = \frac{1}{en\mu_0} \left( \nabla \times \vec{B} \right) \times \vec{B}
\]

\[
\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left[ \frac{1}{en\mu_0} \left( \nabla \times \vec{B} \right) \times \vec{B} \right] \approx -\frac{1}{en\mu_0} \nabla \times \left[ \left( \nabla \times \vec{B} \right) \times \vec{B} \right]
\]

Linearize:

\[
\frac{\partial \vec{B}_1}{\partial t} \approx -\frac{1}{en\mu_0} \nabla \times \left[ \left( \nabla \times \vec{B}_1 \right) \times \vec{B}_0 \right] = -\frac{1}{en\mu_0} \vec{B}_0 \cdot \nabla \left( \nabla \times \vec{B}_1 \right)
\]

Fourier analysis:

\[
\omega \vec{B}_1 \sim \frac{1}{en\mu_0} \vec{B}_0 \cdot \vec{k} \left( \vec{k} \times \vec{B}_1 \right)
\]
Whistler waves: the bad part

\[ \vec{B}_1 \perp \vec{B}_0, \quad \vec{k} \parallel \vec{B}_0 \]

\[ \omega \approx \frac{1}{en \mu_0} B_0 k^2 = \frac{B_0}{\sqrt{\mu_0 n m_i}} \sqrt{\frac{m_i}{e^2 n \mu_0}} k^2 = v_A \frac{c}{\omega_{pi}} k^2 \]

Frequency goes like inverse wavelength squared!

This means that the time step scales like the inverse square of the cell size

Whistlers become important if \( dx \leq \frac{c}{\omega_{pi}} \)
Hall-MHD Computational Cost

Rough estimate: 3D Magnetosphere

Size: $200R_E \times 60R_E \times 60R_E$
Cell size: $dx=0.25R_E$
Without AMR $\sim 4.6 \times 10^7$ cells
Eight MHD state variables
$\sim 4 \times 10^8$ reals, 2-4G bytes

Ignored AMR, structured meshes etc.

Fastest wave mode is fast mode $v_f=(v_A^2+v_S^2)^{1/2}$ or Whistler
Time step limitation is $dt < dx/v_f$, $dx/v_{ph,w}$

Same as MHD as long as ion inertial length not resolved.

Germaschewski, Bhattacharjee
Next step: Multi-fluid models

\[ \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \vec{v}_s) = 0 \]
\[ m_s n_s \frac{\partial}{\partial t} \vec{v}_s + m_s n_s \vec{v}_s \cdot \nabla \vec{v}_s = q_s \left( \vec{E} + \vec{v}_s \times \vec{B} \right) - \nabla p_s \]

\[ \vec{v}_e = \frac{1}{en_e} \left( \sum_s q_s n_s \vec{v}_s - \vec{j} \right) \]
\[ n_e = \frac{1}{e} \sum_s q_s n_s \]
\[ \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \]
\[ \nabla \cdot \vec{B} = 0 \]

+ eqns of state

Continuity
Momentum
Ohm’s law
Electron velocity
Quasi-neutrality
Faraday’s law
Absence of monopoles
Multi-Fluid Computational Cost

Rough estimate: 3D Magnetosphere

Size: $200R_E \times 60R_E \times 60R_E$

Cell size: $dx=0.25R_E$

Without AMR $\sim 4.6 \times 10^7$ cells

$5N+3$ state variables

$\sim 7 \times 10^8$ reals, $3-6G$ bytes (for N=2)

Ignored AMR, structured meshes etc.

Fastest wave mode is fast mode $v_f=(v_A^2+v_S^2)^{1/2}$ or Whistler

Time step limitation is $dt < dx/v_f$, $dx/v_{\text{ph,w}}$, $\Omega_i^{-1}$

Size same as $\sim N \times \text{MHD}$ as long as ion inertial length not resolved. But: time step limited by ion cyclotron frequency.
Kinetic Physics: Hybrid Model

\[
m_s \frac{d\vec{u}_{s,k}}{dt} = q_s \left( \vec{E} + \vec{u}_{s,k} \times \vec{B} \right)
\]

\[
n_s(\vec{r}) = \sum_k S(\vec{r} - \vec{r}_{s,k})
\]

\[
n_s(\vec{r})\vec{v}_s(\vec{r}) = \sum_k S(\vec{r} - \vec{r}_{s,k})\vec{u}_{s,k}
\]

\[
\vec{E} + \vec{v}_e \times \vec{B} = \eta \vec{j} - \frac{1}{en} \nabla p_e - \frac{m_e}{e} \frac{d\vec{v}_e}{dt}
\]

\[
\vec{v}_e = \frac{1}{en_e} \left( \sum_s q_s n_s \vec{v}_s - \vec{j} \right)
\]

\[
n_e = \frac{1}{e} \sum_s q_s n_s
\]

\[
\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}
\]

Eqn of motion of particle \( k \) of species \( s \)

Density

Momentum

Ohm’s law

Electron velocity

Quasi-neutrality

Faraday’s law
Computational effort: Hybrid

Rough estimate: 3D Magnetosphere

Size: $200R_E \times 60R_E \times 60R_E$

Cell size: $dx = c/\omega_{pi} \sim 0.02R_E$ ($\sim 120\text{km}$)

Without AMR $\sim 9 \times 10^{10}$ cells

10 particles/cell, 6 coordinates

$\sim 6 \times 10^{12}$ reals, 24-48Tbytes

Ignored AMR, structured meshes etc.

Additional cost of particle <-> grid operations

Fastest wave mode is Whistler mode $v_{ph,w} = k v_A c/\omega_{pi} \sim 2\pi \lambda^{-1} v_A c/\omega_{pi}$

Time step limitation is $dt < dx/v_{ph,w} \Omega_{ci}^{-1}$
Fully Kinetic Models

Not yet.
How About Transport Models?

- Anisotropic pressure (for regional model)
- Reconnection electric field from kinetic model
- Reconnection electric field from local resistivity (demo project)

… for global magnetosphere
Include Gyrotropic Pressure

\[
\rho \frac{\partial}{\partial t} \vec{v} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla \cdot \vec{p} + \vec{j} \times \vec{B} \]

Momentum

\[
\frac{\partial p_{||}}{\partial t} + \nabla \cdot (p_{||} \vec{v}) + 2 \frac{p_{||}}{B^2} \vec{B} \cdot (\nabla \vec{v}) \cdot \vec{B} - 2 \eta j_{||}^2 = -\frac{2}{3} \frac{p_{||} - p_{\perp} + SB \sqrt{p_{||}}}{\tau_{\text{micro}}} 
\]

\[
\frac{\partial p_{\perp}}{\partial t} + \nabla \cdot (p_{\perp} \vec{v}) + p_{\perp} \nabla \cdot \vec{v} - \frac{p_{\perp}}{B^2} \vec{B} \cdot (\nabla \vec{v}) \cdot \vec{B} - \eta j_{\perp}^2 = \frac{1}{3} \frac{p_{||} - p_{\perp} + SB \sqrt{p_{||}}}{\tau_{\text{micro}}} 
\]

Modified double-adiabatic approach

Hesse and Birn, 1992
Birn et al., 1995
Results

Anisotropy has a big effect on the evolution. How much is right?

Birn et al., 1995
Reconnection Electric Field: What is it?

Electron eqn. of motion:

\[ \vec{E} = -\vec{v}_e \times \vec{B} - \frac{1}{n_e e} \nabla \cdot \vec{P}_e - \frac{m_e}{e} \left( \frac{\vec{\alpha} \vec{v}_e}{\vec{\alpha}} + \vec{v}_e \cdot \nabla \vec{v}_e \right) \]

\[ = -\vec{v}_i \times \vec{B} + \frac{1}{n_e e} \vec{j} \times \vec{B} - \frac{1}{n_e e} \nabla \cdot \vec{P}_e - \frac{m_e}{e} \left( \frac{\vec{\alpha} \vec{v}_e}{\vec{\alpha}} + \vec{v}_e \cdot \nabla \vec{v}_e \right) \]

Key term:

\[ \vec{E} = -\frac{1}{n_e e} \nabla \cdot \vec{P}_e \]
Express Through Ions and Electrons

Basic idea:

\[
\vec{E} = -\vec{v}_i \times \vec{B} + \frac{1}{n_e} \vec{j} \times \vec{B} - \frac{1}{n_e} \nabla \cdot \vec{P}_e
\]

\[
\approx -\vec{v}_i \times \vec{B} + \frac{1}{n_i} \nabla \cdot \vec{P}_i
\]

Analytic theory:

\[
E_{rec} \approx \frac{1}{e} \sqrt{2m_eT_e} \frac{\partial v_{e,x}}{\partial x}
\]

Can ion pressure tensor represent kinetic physics?

Test with equal ion and electron mass…
Kinetic, $m_i=m_e$ Simulation
Reconnection rate: Fast enough?
Reconnection electric field: Where is it?

\[ E_y = -\frac{1}{n_e e} \left( \frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yz}}{\partial z} \right) - \frac{m_e}{e} \vec{v}_e \cdot \nabla \vec{v}_y \]

- Strongly localized dissipation region, scales smaller than CS length
- Expressed by ion pressure tensor
Implementation: How to make MHD fly

Two avenues:

- Use kinetic theory results to model ion pressure tensor

- Design a resistivity based on kinetic results
1. Match Kinetic Model to MHD

\[ E_{\text{REC}} \approx B_Z (d_x) V_x (d_x) \sim B'_Z V'_x d^2_x = \frac{1}{e} \left(\frac{2 m_i}{T_i}\right)^{1/2} \frac{\partial V_x}{\partial x} \]

MHD Region
\[ E = -[V \times B] \]

\[ d_x = \left(\frac{V_{T_i} m_i}{e B'_Z}\right)^{1/2} \]

\[ d_z = \left(\frac{V_{T_i} m_i}{e B_x'}\right)^{1/2} \]

Y axis is \parallel to \mathbf{J}
1) Search for reconnection sites: $X_0(y)$, $X_1(y)$

2) Calculate spatial scales of diffusion regions and reconnection electric field

$$d_x^2 = (\alpha / B_z) \left( \frac{2P}{\rho} \right)^{1/2}$$

$$d_z^2 = (\alpha / B_x) \left( \frac{2P}{\rho} \right)^{1/2}$$

$$E_{REC}^2 = \alpha \left( \frac{2P}{\rho} \right)^{1/2} \frac{\partial V_x}{\partial x}$$

$$\alpha = \rho_{i0} / R_E$$

3) Add non-gyrotropic correction to the induction equation

$$E_{Y}^{NG}(x,y,z) = \frac{E_{REC}(y)}{\cosh \left[ \frac{(x-X_0(y))}{d_x(y)} \right] \cosh \left[ \frac{z}{d_z(y)} \right]}$$
Solar Wind Parameters:
N = 2 cm$^{-3}$, T = 20000 K, Vx = -400 km/s, |B| = 10 nT

Simulation
Detailed Comparison

Nongyrotropic Corrections

Numerical Resistivity

Kinetic Simulations

Resistive MHD Simulations

M. Kuznetsova
2. Calculate Equivalent Resistivity

minimum (equivalent) Lundquist number: \( R = \frac{\mu_0 v_A L}{\eta} = 3 \)

localization: \( \delta x = 25, \delta z = 2.5 \)

\[ \eta = \frac{E_{\text{reconnection}}}{j_y} \]
MHD Simulation with Local Resistivity
MHD Reconnection Rate

Fast reconnection possible with suitable resistivity
MHD will remain the tool of choice for large scale modeling

- Considering MHD approximations, MHD is a extremely successful research and quantitative modeling tool

- Beyond changing models altogether, kinetic physics can be incorporated into MHD models through
  -- Suitable transport models
  -- Modifications of the equations

- Much of this work is in experimental stage, but shows promising results
  -- MHD can reproduce fast reconnection

- Other (related) avenues of success:
  -- Local overriding MHD by better model: MHD/RCM coupling
  -- Data assimilation (how?)
  -- Sub-gridscale modeling

Adding physics exacts a price, but that price may be well worth paying.
Final Look

Engaged

..but not married - yet